ON GENERIC VANISHING FOR PLURICANONICAL BUNDLES

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1. Introduction

Thoughout this article, we work over the complex number field.

Generic vanishing theory is the study of the family of the cohomologies

$$\{H^i(X, \mathcal{F} \otimes \mathcal{L}) | \mathcal{L} \in \mathrm{Pic}^0(X)\}$$

of a fixed coherent sheaf $\mathcal{F}$. Let $f : X \to A$ be a morphism from a smooth projective variety X to an abelian variety A and $\mathcal{F}$ a coherent sheaf on X. We take a closed subset

$$V^i_k(\mathcal{F}, f) = \{\alpha \in \mathrm{Pic}^0(A) | h^i(X, \mathcal{F} \otimes f^*\alpha) \geq k\}$$

of $\mathrm{Pic}^0(A)$ for integers $i, k \geq 0$. Our aim is to investigate the dimensions and structures of $V^i_k(\mathcal{F}, f)$. If f is the Albanese morphism, $V^i_k(\mathcal{F}, f)$ is denoted by $V^i_k(\mathcal{F})$. Moreover we also denote $V^i_1(\mathcal{F}, f)$ by $V^i(\mathcal{F}, f)$.

For $\mathcal{F} = \omega_X$, we have the following good results.

Theorem 1.1 (The generic vanishing theorem, Green–Lazarsfeld [GrLa])**.** *Let $f : X \to A$ be a morphism from a smooth projective variety X to an abelian variety A. Then* $\operatorname{codim} V^i(\omega_X, f) \geq i - (\dim X - \dim f(X))$ *for every $i \geq 1$.*

Definition 1.2 (GV-sheaf)**.** Let A be an abelian variety and $\mathcal{F}$ a coherent sheaf on A. $\mathcal{F}$ is called a *generic vanishing sheaf* (*GV-sheaf*, for short) if $\operatorname{codim} V^i(\mathcal{F}) \geq i$ for every $i \geq 1$.

Theorem 1.3 (Hacon [Hac])**.** *Let $f : X \to A$ be a morphism from a smooth projective variety X to an abelian variety A. Then $R^j f_*\omega_X$ is a GV-sheaf for every $j \geq 0$.*

Remark 1.4. Theorem 1.3 implies Thm 1.1 by using Kollár's theorem on the higher direct images of ω_X.

Definition 1.5 (Torsion subvariety)**.** Let A be an abelian variety and T a closed subvariety of A. T is called a *torsion subvariety* if T is a translate of an abelian subvariety by a torsion point.

Theorem 1.6 (Simpson [Sim]). *Let $f : X \to A$ be a morphism from a smooth projective variety X to an abelian variety A. Then $V^j_k(\omega_X, f)$ is a finite union of torsion subvarieties of* $\mathrm{Pic}^0(A)$ *for every $j \geq 0$ and $k \geq 1$.*

We investigate whether these theorems can be generalized to log pluricanonical bundles $\mathcal{O}_X(m(K_X + \Delta))$ of a log canonical pair (X, Δ).

2. Main results

We fix the following notation and convention.

- X is a smooth projective variety, A is an abelian variety, and $f : X \to A$ is a morphism.
- Δ is a boundary $\mathbb{Q}$-divisor on X with simple normal crossing support, that is, a $\mathbb{Q}$-divisor on X whose coefficients are in $[0, 1]$ and $\mathrm{Supp}\Delta$ is a simple normal crossing divisor.
- When considering a $\mathbb{Q}$-divisor $m(K_X + \Delta)$ for some positive integer m, we always assume that there exists a Cartier divisor D on X such that $D \sim_{\mathbb{Q}} m(K_X + \Delta)$. So we can consider $V^j_k(m(K_X + \Delta), f)$ and $R^j f_*\mathcal{O}_X(m(K_X + \Delta))$.
- H^j_m denotes the statement that $R^j f_*\mathcal{O}_X(m(K_X + \Delta))$ is a GV-sheaf on A, which is a generalization of Theorem 1.3.
- S^j_m denotes the statement that $V^j_k(m(K_X + \Delta), f)$ is a finite union of torsion subvarieties for every $k \geq 1$, which is a generalization of Theorem 1.6.

Theorem 2.1. *In the above notation, the following hold.*

(i) H^j_1 *holds for every* $j \geq 0$.
(ii) H^0_m *holds for every* $m \geq 1$.
(iii) *If* $j \geq 1$ *and* $m \geq 2$, *then* H^j_m *does not hold in general.*
(iv) S^j_1 *holds for every* $j \geq 0$.
(v) S^0_m *holds for every* $m \geq 1$.
(vi) *If* $j \geq 1$ *and* $m \geq 2$, *then* S^j_m *does not hold in general.*

Remark 2.2.

- (ii) is a result of Popa–Schnell [PoSc, Theorem 1.10]. (i) is immediately deduced by their argument, although it is not explicitly stated in their paper. So (i) and (ii) are not new results.
- The KLT case of (iv) was proved by Clemens–Hacon [ClHa, Theorem 8.3].
- The $\Delta = 0$ case of Theorem 2.1 (v) was proved by Chen–Hacon [ChHa, Theorem 3.2].

- (ii) and (v) holds for general projective log canonical pairs, which are easily reduced to the log smooth case by taking a log resolution.
- For (iii) and (vi), we will construct counterexamples. In fact, those counterexamples can be taken as $\Delta = 0$ (just a pluri-canonical bundle, not a log pluricanonical bundle).

We can generalize the generic vanishing theorem for log canonical pairs by using Theorem 2.1 (i).

Theorem 2.3. *Let X be a smooth projective variety, Δ a boundary $\mathbb{Q}$-divisor on X with simple normal crossing support, $f : X \to A$ a morphism to an abelian variety, and D a Cartier divisor on X such that $D \sim_{\mathbb{Q}} K_X + \Delta$. Set $l = \max\{\dim V - \dim f(V) \mid V = X \text{ or } V \text{ is a log canonical center of } (X, \Delta)\}$. Then*

$$\operatorname{codim} V^i(D, f) \geq i - l$$

for any i.

Proof. Set $S = \lfloor \Delta \rfloor$ and let Δ_i be an irreducible component of S. Consider the exact sequence $\cdots \to R^j f_* \mathcal{O}_X(D - \Delta_i) \to R^j f_* \mathcal{O}_X(D) \to R^j f_* \mathcal{O}_{\Delta_i}(D|_{\Delta_i}) \to \cdots$. Then it follows that $R^j f_* \mathcal{O}_X(D) = 0$ for $j > l$ by induction of both the dimension of X and the number of irreducible components of S.

Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(A, R^q f_* \mathcal{O}_X(D) \otimes \xi) \Rightarrow H^{p+q}(X, \mathcal{O}_X(D) \otimes f^* \xi),$$

where $\xi \in \operatorname{Pic}^0(A)$. Then it follows by the spectral sequence that

$$V^i(D, f) \subset \bigcup_{q=0}^{l} V^{i-q}(R^q f_* \mathcal{O}_X(D)).$$

Furthermore, $R^q f_* \mathcal{O}_X(D)$ are GV-sheaves on A for all q by Theorem 2.1 (i), so

$$\operatorname{codim} V^{i-q}(R^q f_* \mathcal{O}_X(D)) \geq i - q$$

for $0 \leq i \leq l$. Hence $\operatorname{codim} V^i(D, f) \geq i - l$. □

Proof of Theorem 2.1 (iii). We will construct an irregular smooth projective variety of dimension ≥ 2 with big anti-canonical bundle and show that such a variety does not satisfy H^j_m for some $j \geq 1$ and $m \geq 2$.

Let A be an abelian variety. We take an ample line bundle L on A and define a vector bundle E as the direct sum of L^{-1} and $\mathcal{O}_A$. Let $\pi : X = \mathbb{P}_A(E) \to A$ be the projective bundle on A associated to E. Clearly the irregularity $q(X)$ of X is positive. The canonical bundle

ω_X is isomorphic to $\pi^*(\omega_A \otimes \det E) \otimes \mathcal{O}_X(-\operatorname{rank} E) = \pi^* L^{-1} \otimes \mathcal{O}_X(-2)$ (see [Laz, 7.3.A]).

We will see that ω_X^{-1} is big. Let ξ and l be the numerical classes of $\mathcal{O}_X(1)$ and L, respectively. Note that ξ is an effective class since $H^0(X, \mathcal{O}_X(1)) = H^0(A, E) \neq 0$. The numerical class of ω_X^{-1} is equal to

$$2\xi + \pi^* l = \frac{N-1}{N} 2\xi + \frac{1}{N} 2\xi + \pi^* l,$$

where N is a sufficiently large integer such that $(1/N)2\xi + \pi^* l$ is ample. So the numerical class of ω_X^{-1} is represented by the sum of an effective class and an ample class. Therefore ω_X^{-1} is big.

Let $f = \mathrm{alb}_X : X \to A$ be the Albanese morphism of X. Now we show that $R^j f_* \omega_X^{\otimes m}$ is not a GV-sheaf for some positive integers j and m.

Now we prove the following lemma.

Lemma 2.4. *Let X be a smooth projective variety of dimension n and D be a big Cartier divisor on X. Then*

$$V^0(mD) = \{\xi \in \operatorname{Pic}^0(X) \mid H^0(X, \mathcal{O}_X(mD + \xi)) \neq 0\} = \operatorname{Pic}^0(X)$$

for any sufficiently large and divisible m.

Proof. Since D is big, there exist a positive integer m_0, a very ample Cartier divisor H, and an effective Cartier divisor E such that $m_0 D \sim H + E$. For any positive integer m, we have

$$V^0(mm_0D) = V^0(mH + mE) \supset V^0(mH).$$

We can take a positive integer m_1 satisfying that

$$H^i(X, \mathcal{O}_X(mH + \xi)) = 0$$

for every $\xi \in \operatorname{Pic}^0(X)$, $m \geq m_1$ and $i > 0$ (take m_1 such that $m_1 H - K_X$ is ample). According to the notion of the Castelnuovo–Mumford regularity, $mH + \xi$ is 0-regular for every $\xi \in \operatorname{Pic}^0(X)$ and $m \geq m_1 + n$, and so it is globally generated. In particular, $V^0(mH) = \operatorname{Pic}^0(X)$ for every $m \geq m_1 + n$. Therefore $V^0(mm_0D) = \operatorname{Pic}^0(X)$ for every $m \geq m_1 + n$. $\square$

By the above lemma, we can take a positive integer m such that $V^n(\omega_X^{\otimes m}, f) = -V^0(\omega_X^{\otimes(1-m)}, f) = \operatorname{Pic}^0(A)$. Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(A, R^q f_* \omega_X^{\otimes m} \otimes \alpha) \Rightarrow H^{p+q}(X, \omega_X^{\otimes m} \otimes f^*\alpha), \quad \alpha \in \operatorname{Pic}^0(A).$$

Then it follows that

$$\mathrm{Pic}^0(A) = V^n(\omega_X^{\otimes m}, f) \subset \bigcup_{i=0}^{n} V^i(R^{n-i} f_* \omega_X^{\otimes m}).$$

So $V^i(R^{n-i} f_* \omega_X^{\otimes m}) = \mathrm{Pic}^0(A)$ for some i. Note that $i > 0$ since $R^n f_* \omega_X^{\otimes m} = 0$. Hence it follows that $R^{n-i} f_* \omega_X^{\otimes m}$ is not a GV-sheaf. □

Theorem 2.1 (ii) is proved by the following vanishing theorem.

Theorem 2.5 (Popa–Schnell [PoSc, Theorem 1.7])**.** *Let (X, Δ) be a projective log canonical pair, Y a projective variety, $g : X \to Y$ a morphism, andL an ample and globally generated line bundle on Y. Take an integer $m \geq 1$. Then $H^i(Y, f_* \mathcal{O}_X(m(K_X + \Delta)) \otimes L^{\otimes l}) = 0$ for every $i > 0$ and $l \geq (m-1)(\dim Y + 1) + 1$.*

Conversely, by a similar argument, Theorem 2.1 (iii) implies that the above vanishing does not hold for higher cohomologies of pluricanonical bundles in general.

Corollary 2.6. *Let X be a smooth projective variety, Y a projective variety, $g : X \to Y$ a morphism, and L an ample and globally generated line bundle on Y. Take integers $j \geq 1$ and $m \geq 2$. Then we can not take a positive integer $N = N(j, m, \dim Y)$ depending only on j, m and $\dim Y$ such that $H^i(Y, f_* \omega_X^{\otimes m} \otimes L^{\otimes l}) = 0$ for every $i > 0$ and $l \geq N$.*

Sketch of proof of Theorem 2.1 (iv). (For a detailed proof, see [Shi, Theorem 3.5].) By assumption, there exists a Cartier divisor D such that $D \sim_{\mathbb{Q}} K_X + \Delta$. Set $C = D - (K_X + \lfloor \Delta \rfloor)$. Since $C \sim_{\mathbb{Q}} \{\Delta\}$, $NC \sim N\{\Delta\}$ for some positive integer N. Take the normalization of the cyclic cover $\mathrm{Spec} \bigoplus_{k=0}^{N-1} \mathcal{O}_X(-kC) \to X$. Then

$$\pi_* \mathcal{O}_Y = \bigoplus_{k=0}^{N-1} \mathcal{O}_X(-kC + \lfloor k\{\Delta\} \rfloor).$$

So $\pi_* \mathcal{O}_Y(-\pi^* \lfloor \Delta \rfloor)$ contains $\mathcal{O}_X(-C - \lfloor \Delta \rfloor) = \mathcal{O}_X(-(D - K_X))$ as a direct summand. Hence it follows that, if $V_k^j(-\pi^* \lfloor \Delta \rfloor, f \circ \pi)$ is a finite union of torsion subvarieties for every j and k, then $V_k^j(D, f)$ is also a finite union of torsion subvarieties for every j and k. Further, we can show that $(Y, \pi_* \lfloor \Delta \rfloor)$ is a log canonical pair.

Take a log resolution $\mu : Y' \to Y$ of $(Y, \pi^* \lfloor \Delta \rfloor)$. Set

$$\Delta_{Y'} = \mu^*(K_Y + \pi^* \lfloor \Delta \rfloor) - K_{Y'},$$

then

$$\Delta_{Y'}^{=1} = \mu_*^{-1}(\pi^* \lfloor \Delta \rfloor) + E$$

for some reduced μ-exceptional divisor E. Since $-K_{Y'/Y} = -(K_{Y'} - \mu^* K_Y)$ has no irreducible components with coefficient 1, every component of E is contained in $\mu^*\pi^*\lfloor\Delta\rfloor$. Since E is μ-exceptional, E is in fact contained in $\mu^*\pi^*\lfloor\Delta\rfloor - \mu_*^{-1}\pi^*\lfloor\Delta\rfloor$. So $F = \mu^*\pi^*\lfloor\Delta\rfloor - \mu_*^{-1}\pi^*\lfloor\Delta\rfloor - E$ is an effective and μ-exceptional divisor on Y'. By the Fujino–Kovács vanishing theorem (see [Kov] and [Fuj2]),

$$R^i\mu_*\mathcal{O}_{Y'}(-\Delta_{Y'}^{=1}) = 0$$

for $i > 0$. Therefore

$$\begin{aligned} R\mu_*\mathcal{O}_{Y'}(-\Delta_{Y'}^{=1}) &\cong \mu_*\mathcal{O}_{Y'}(-\Delta_{Y'}^{=1}) \\ &\cong \mu_*\mathcal{O}_{Y'}(-\mu_*^{-1}\pi^*\lfloor\Delta\rfloor - E) \\ &\cong \mu_*\mathcal{O}_{Y'}(-\mu^*\pi^*\lfloor\Delta\rfloor + F) \\ &\cong \mathcal{O}_Y(-\pi^*\lfloor\Delta\rfloor). \end{aligned}$$

Thus we have $V_k^j(-\Delta_{Y'}^{=1}, f\circ\pi\circ\mu) = V_k^j(-\pi^*\lfloor\Delta\rfloor, f\circ\pi)$. Moreover, $\Delta_{Y'}^{=1}$ is a simple normal crossing divisor on Y'. Then the proof is reduced to the case when Δ is a simple normal crossing divisor. This case holds due to Budur [Bud]. □

Sketch of proof of Theorem 2.1 (v). (For a detailed proof, see [Shi, Theorem 3.9].) Take any point $\xi \in V_k^0(m(K_X + \Delta), f)$. Then there exists $\xi_0 \in \mathrm{Pic}^0(A)$ such that $\xi = m\xi_0$. After replacing (X, Δ) by a suitable log resolution, we can take a Cartier divisor D_0 on X such that

- $D_0 \sim_{\mathbb{Q}} K_X + \Delta_0$, where Δ_0: a boundary $\mathbb{Q}$-divisor with SNC support,
- $\xi_0 \in V_k^0(D_0, f)$, and
- $V_k^0(D_0, f) + (m-1)\xi_0 \subset V_k^0(m(K_X + \Delta), f)$.

By Theorem 2.1 (iv), $V_k^0(D_0, f)$ is a finite union of torsion subvarieties. So there exist an abelian subvariety B of A and a torion point q of A such that $\xi_0 \in B + q \subset V_k^0(D_0, f)$. Then

$$\xi = \xi_0 + (m-1)\xi_0 \in B + q + (m-1)\xi_0 \subset V_k^0(m(K_X + \Delta), f).$$

Since $\xi_0 \in B + q$, $\xi_0 = b + q$ for some $b \in B$. So

$$B + q + (m-1)\xi_0 = B + q + (m-1)b + (m-1)q = B + mq.$$

Therefore

$$\xi \in B + mq \in V_k^0(m(K_X + \Delta), f).$$

So $V_k^0(m(K_X + \Delta), f)$ is a union of torsion subvarieties. Since the set of torsion subvarieties of A is countable, $V_k^0(m(K_X + \Delta), f)$ is in fact a finite union of torsion subvarieties. □

Corollary 2.7 (Campana–Koziarz–Păun [CKP], Kawamata [Kaw]). *Let (X, Δ) be a projective log canonical pair. Assume that $K_X + \Delta \equiv 0$. Then $K_X + \Delta \sim_{\mathbb{Q}} 0$.*

We give another proof of this theorem.

Proof. By taking a log resolution, we may assume that (X, Δ) is log smooth. Take $m > 0$ such that $\alpha = m(K_X + \Delta) \in \mathrm{Pic}^0(X)$. Then $h^0(X, m(K_X + \Delta) - \alpha) = h^0(X, \mathcal{O}_X) \neq 0$, so $-\alpha \in V^0(m(K_X + \Delta))$. Hence $V^0(m(K_X + \Delta))$ is non-empty. Theorem 2.1 (v) implies that there exists a torsion point $\beta \in V^0(m(K_X + \Delta))$. This means that $m(K_X + \Delta) \sim_{\mathbb{Q}} 0$. □

In addition, we give the following corollary, which is an implication of Iitaka's subadditivity conjecture.

Corollary 2.8. *Let (X, Δ) be a projective log canonical pair, A an abelian variety, $f : X \to A$ a surjective morphism with connected fibers, and F a sufficiently general fiber of f. Assume that $\kappa((K_X+\Delta)|_F) \geq 0$. Then $\kappa(K_X + \Delta) \geq 0$.*

Proof. Take $m > 0$ such that $m(K_X + \Delta)$ is Cartier and $h^0(m(K_X + \Delta)|_F) \neq 0$. Then $f_*\mathcal{O}_X(m(K_X + \Delta)) \neq 0$. Theorem 2.1 (ii) implies that $f_*\mathcal{O}_X(m(K_X + \Delta))$ is a GV-sheaf on A.

Now we need the fact that, for a GV-sheaf $\mathcal{F}$ on A, $\mathcal{F} \neq 0$ if and only if $V^0(\mathcal{F}) \neq \varnothing$. So $V^0(f_*\mathcal{O}_X(m(K_X+\Delta))) \neq \varnothing$. Then Theorem 2.1 (v) implies that there exists a torsion point $\alpha \in V^0(f_*\mathcal{O}_X(m(K_X + \Delta)))$. Take $N > 0$ such that $N\alpha = 0$. We compute

$$\begin{aligned} h^0(X, \mathcal{O}_X(Nm(K_X + \Delta))) &= h^0(X, \mathcal{O}_X(Nm(K_X + \Delta)) \otimes f^*\alpha^{\otimes N}) \\ &\geq h^0(X, \mathcal{O}_X(m(K_X + \Delta)) \otimes f^*\alpha) \\ &= h^0(A, f_*\mathcal{O}_X(m(K_X + \Delta)) \otimes \alpha) \\ &\neq 0. \end{aligned}$$

So $\kappa(K_X + \Delta) \geq 0$. □

Proof of Theorem 2.1 (vi). Let E be an elliptic curve, L a principal polarization on E (i.e. an ample line bundle on E with $h^0(L) = 1$), and A an abelian variety of dimension $g \geq 2$ including E as a proper abelian subvariety. Take a non-torsion point $a \in A$ and define a closed immersion $\iota : E \to A$ by $\iota(x) = x + a$. By definistion, $\iota(E) = E + a$. Let $\hat{A}$ be the dual abelian variety of A, and $R\Phi : D(A) \to D(\hat{A})$ and $R\Psi : D(\hat{A}) \to D(A)$ the Fourier–Mukai transforms.

Set $F = R\Phi\iota_*L \in D(\hat{A})$. Since $h^i(E, L \otimes \iota^*\alpha) = 0$ for $i > 0$ and $\alpha \in \mathrm{Pic}^0(A)$ by Kodaira vanishing, $R^i\Phi\iota_*L = 0$ for $i > 0$. So $F =$

$\Phi\iota_*L$. Furthermore $h^0(E, L\otimes\iota^*\alpha) = \chi(E, L\otimes\iota^*\alpha) = \chi(E, L) = 1$ for $\alpha \in \mathrm{Pic}^0(A)$, so F is in fact a line bundle on $\hat{A}$.

Take a vector bundle $V = F \oplus \mathcal{O}_{\hat{A}}$ on $\hat{A}$. Let $\pi : X = \mathbb{P}_{\hat{A}}(V) \to \hat{A}$ be the projective bundle over $\hat{A}$ associated to V. Then

$$\begin{aligned}\omega_X &= \pi^*(\omega_{\hat{A}} \otimes \det V) \otimes \mathcal{O}_{\hat{A}}(-\operatorname{rank} V)\\ &= \pi^*F \otimes \mathcal{O}_{\hat{A}}(-2)\end{aligned}$$

(cf. [Laz, 7.3.A]). Therefore

$$\begin{aligned}\pi_*(\omega_X^{-1}) &= F^{-1} \otimes \pi_*\mathcal{O}_{\hat{A}}(2)\\ &= F^{-1} \otimes S^2V\\ &= F^{-1} \otimes (F^2 \oplus F \oplus \mathcal{O}_{\hat{A}})\\ &= F \oplus \mathcal{O}_{\hat{A}} \oplus F^{-1}.\end{aligned}$$

So

$$\begin{aligned}V^{g+1}(\omega_X^2) = V^{g+1}(\omega_X^2, \pi) &= -V^0(\omega_X^{-1}, \pi) = -V^0(\pi_*\omega_X^{-1})\\ &= -V^0(F) \cup -V^0(\mathcal{O}_{\hat{A}}) \cup -V^0(F^{-1})\\ &= V^g(F^{-1}) \cup \{0\} \cup V^g(F)\end{aligned}$$

(note that π is the Albanese morphism of X, so we have the first equality).

First we calculate $V^g(F^{-1})$.

$$\begin{aligned}V^g(F^{-1}) &= \{a \in A | h^g(\hat{A}, F^{-1} \otimes L_a) \neq 0\}\\ &= \{a \in A | h^g(\hat{A}, (-1)^*(F^{-1} \otimes L_a)) \neq 0\}\\ &= \{a \in A | h^g(\hat{A}, (-1)^*F^{-1} \otimes L_{-a}) \neq 0\}\\ &= -V^g((-1)^*F^{-1})\\ &= -\mathrm{Supp} R^g\Psi(-1)^*F^{-1},\end{aligned}$$

where $(-1) : \hat{A} \to \hat{A}$ is the multiplication by -1. The last equation follows by the base change theorem. We write $R\Delta(\cdot) = R\mathscr{H}om(\cdot, \mathcal{O}_{\hat{A}})$. Then

$$\begin{aligned}R\Psi(-1)^*F^{-1} &= R\Psi(-1)^*R\Delta R\Phi\iota_*L\\ &= R\Psi(-1)^*(-1)^*R\Phi R\Delta\iota_*L[g] \quad (R\Phi R\Delta(\cdot) = (-1)^*R\Phi R\Delta(\cdot)[g])\\ &= (-1)^*R\Delta\iota_*L \quad \text{(by Mukai's theorem)}\\ &= (-1)^*R\mathscr{H}om(\iota_*L, \mathcal{O}_{\hat{A}})\\ &= (-1)^*R\iota_*R\mathscr{H}om(L, \mathcal{O}_{\hat{A}} \otimes \omega_{E/A}[1-g]) \quad \text{(Grothendieck duality)}\\ &= (-1)^*R\iota_*L^{-1}[1-g].\end{aligned}$$

So $R^g\Psi(-1)^*F^{-1} = (-1)^*R^1\iota_*L^{-1} = 0$. This implies that $V^g(F^{-1}) = -V^g((-1)^*F^{-1}) = \varnothing$ (using base change theorem).

Next, we calculate $V^g(F)$. By base change theorem, $V^g(F) = \mathrm{Supp}R^g\Psi F$. We have $R^g\Psi F = R^g\Psi R\Phi\iota_*L = (-1)^*\iota_*L$ by Mukai's theorem, so

$$V^g(F) = \mathrm{Supp}R^g\Psi F = (-1)^{-1}(\mathrm{Supp}\iota_*L) = E - a.$$

Consequently, we have

$$V^{g+1}(\omega_X^2) = \{0\} \cup E - a.$$

Therefore $V^{g+1}(\omega_X^2)$ is not a union of torsion translates. □

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